Can We Measure the Complexity of Mathematical Statements?

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Auckland, March 2010



Main question

Can we compare in a reasonable way the complexity of

- the four colour theorem,
- Fermat's last theorem,
- the Riemann hypothesis?



More questions

An affirmative answer to this question ought to be based on a "mathematical property" shared by all statements we wish to compare...

Is there a natural candidate for such a "mathematical property"?



A “shared mathematical property”

We propose the property that a statement is **finitely refutable**, i.e. the statement can be refuted by a single counter-example.

For example, Fermat’s last theorem, which states that there are no positive integers x, y, z satisfying the equation $x^n + y^n = z^n$, for any integer value $n > 2$, is finitely refutable because a single positive integer-valued vector $(x, y, z, n), n > 2$ for which the equality is $x^n + y^n = z^n$ is true, refutes the statement.

Π_1–problems

A problem π of the form

$$\forall k P(k),$$

where P is a computable predicate is called a Π_1–*problem*.

Any Π_1–problem is finitely refutable.

Fermat's last theorem is a Π_1–problem. The predicate $P(k)$ is true iff

a) $k = J(J(J(n, x), y,)z)$ and
b) $J(a, b) = (a + b)(a + b + 1)/2$ and
c) $n > 2$ and
d) $x, y, z > 0$ and
e) $x^n = y^n + z^n$.

What about the four colour theorem and the Riemann hypothesis?

A uniform way to solve Π_1–problems

Fix U a universal Turing machine.

For every Π_1–problem $\pi = \forall k P(k)$ we associate a program for U which computes the first counter-example (if any):

$$\sigma_\pi =_{\mathrm{def}} \inf\{k \,:\, P(k) = \text{ false}\}.$$

This program satisfies:

$$\pi \text{ is true iff } U(\sigma_\pi) = \infty.$$



A universal Turing machine

A simple, minimal universal Turing machine U can be designed using the following five instructions:

= r1 r2 r3
& r1 r2
+ r1 r2
! r1
%

An example of a program for U

The following program computes in d the product of two non-negative integers a and b:

number	instruction
0	`&h,e`
1	`&d,0`
2	`=b,0,8`
3	`&e,1`
4	`+d,a`
5	`=b,e,8`
6	`+e,1`
7	`=a,a,4`
8	`&e,h`
9	`=a,a,c`



The halting problem

The **halting problem** for a Turing machine V is the function Λ_V defined for every program σ by

$$halt_V(\sigma) = \begin{cases} 0, & \text{if } V(\sigma) = \infty, \\ 1, & \text{if } V(\sigma) < \infty\,. \end{cases}$$

Turing's undecidability theorem. The halting problem for a **universal** machine U

$$halt_U(\sigma) = \begin{cases} 0, & \text{if } U(\sigma) = \infty, \\ 1, & \text{if } U(\sigma) < \infty\,, \end{cases}$$

is incomputable (by any Turing machine).

A complexity measure

So, we have a uniform way to solve every Π_1–problem which is . . . useless.

Not completely! We can introduce a complexity measure of a Π_1–problem by computing the size of the smallest program searching for a counter-example to the problem:

The complexity (with respect to U) of a Π_1–problem π is defined by

$$C_U(\pi) = \inf\{|\Pi_P| : \pi = \forall k P(k)\},$$

where inf is taken according to all possible P's and Π_P's.

Computing the size of a program

number	instruction	code	length
0	`&h,e`	01 0001001 00110	14
1	`&d,0`	01 00101 100	10
2	`=b,0,8`	00 011 100 1110010	15
3	`&e,1`	01 00110 101	10
4	`+d,a`	111 00101 010	11
5	`=b,e,8`	00 011 00110 1110010	17
6	`+e,1`	111 00110 101	11
7	`=a,a,4`	00 010 010 11010	13
8	`&e,h`	01 00110 0001001	14
9	`=a,a,c`	00 010 010 00100	13

Total length: 128.

Estimating the complexity

The complexity C_U is incomputable, so the goal is to compute an upper bound of the complexity $C_U(\pi)$ by choosing a representation $\pi = \forall k P(k)$ for which $|\Pi_P|$ is the smallest possible.

The running time efficiency of the program Π_P is irrelevant here, the size in bits counts.

According to the complexity C_U we classify Π_1–problems into the following classes:

$$\mathfrak{C}_{U,n} = \{\pi : \pi \text{ is a } \Pi_1\text{–problem}, C_U(\pi) \leq n \text{ kbit}\}.$$

Remarkable Π_1–problems

Theorem. The four colour theorem is a Π_1–problem.

Theorem. Riemann hypothesis is a Π_1–problem.

Not all problems are Π_1–problems. For example, the twin-prime conjecture is not a Π_1–problem.



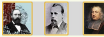
Riemann hypothesis predicate

The negation of the Riemann hypothesis is equivalent to the existence of positive integers k, l, m, n satisfying the following:

1. $n \geq 600$,
2. $\forall y < n\,[(y+1) \mid m]$,
3. $m > 0\,\&\,\forall y < m\,[y = 0 \vee \exists x < n\,[\neg\,[(x+1) \mid y]]]$,
4. $explog(m-1, l)$,
5. $explog(n-1, k)$,
6. $(l-n)^2 > 4n^2k^4$,

where $x \mid z$ means "x divides z" and $explog(a, b)$ is the predicate

$$\exists x\,[x > b+1\,\&\,(1+1/x)^{xb} \leq a+1 < 4(1+1/x)^{xb}].$$



Computational results

According to U we have classified some mathematical statements:

1. Legendre's conjecture (for any natural number $n > 1$ there exists a prime number p such that $n^2 \leq p \leq (n+1)^2$) (416), Fermat's last theorem (738) and Goldbach's conjecture (every even integer greater than 2 can be expressed as the sum of two primes) (756) are in $\mathfrak{C}_{U,1}$,
2. Dyson's conjecture (the reverse of a power of two is never a power of five) (1067) is in $\mathfrak{C}_{U,2}$,
3. the Riemann hypothesis (2741) is in $\mathfrak{C}_{U,3}$,
4. the four colour theorem (3289) is in $\mathfrak{C}_{U,4}$.

We proved that Collatz's $3x+1$ problem is a Π_1–problem, but the proof is not constructive, so we could not (yet) classify it.

Open questions

Rank according to complexity the following problems:

- The formula: $e^{i\pi} = -1$,
- The formula: $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \pi^2/6$,
- The statement: “At any party, there is a pair of people who have the same number of friends present,”
- The statement: “π is transcendental.”



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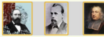

A Turing machine