Circuits and Attractors in Kauffman Networks

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Abstract

There has been some ambiguity about the growth of attractors in Kauffman networks with network size. Some recent work has linked this to the role and growth of circuits or loops of boolean variables. Using numerical methods we have investigated the growth of structural circuits in Kauffman networks and suggest that the exponential growth in the number of structural circuits places a lower bound on the complexity of the growth of boolean dependency loops and hence of the number of attractors. We use a fast and exact circuit enumeration method that does not rely on sampling trajectories. We also explore the role of structural self-edges, or self-inputs in the NK-model, and how they affect the number of structural circuits and hence of attractors.

Keywords: Kauffman networks; random boolean functions; circuit enumeration; loops; attractors; complexity.

Introduction

Kauffman networks have N nodes each with a boolean bit state and a K-input boolean function. Each node is connected to K other nodes, so the resulting directed graph has each node with exactly K inputs but with a variable number of outputs. If nodes' inputs are chosen uniformly, the outputs per node form a Poisson distribution.

The nodes are assigned a random bit state and one of the 2^{2^K} possible boolean functions of K inputs. The network is then updated synchronously so that each node's new state is determined by the inputs presented to its boolean function.

For certain values of K the network quickly arrives at a fixed stable state. Above a critical value of $K = 2$, the network remains in a chaotic state with different periodic oscillations (attractors) forming. It has remained a mystery why the number of attractors grows so rapidly, but our work has shed light on this by exactly enumerating the number of structural circuits or loops that can form in the network.

The static number of circuits has to be a lower bound on the number of attractors in the dynamical system, and exact enumeration shows that the number of circuits in the graph grows exponentially with the network size N.

Kauffman networks and related systems are thought to model gene regulatory networks, whereby one gene (node) controls the expression of another in an organism's genotype.

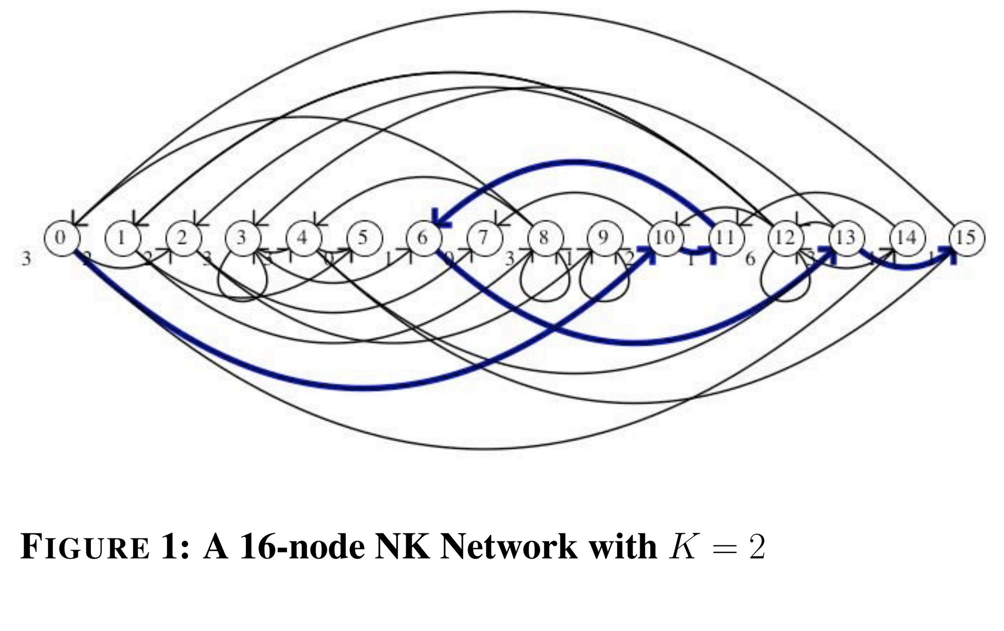

FIGURE 1: A 16-node NK Network with $K = 2$

Methods

We have developed an algorithm to count exactly the circuits in a directed graph.

Exact enumeration (as shown in figure 2) indicates there are 22 circuits composed as follows: 4 of length 1; 2 of length 3; 4 of length 5; 2 of length 6; 6 of length 8; and 4 of length 10. If self-edges are disallowed we would obtain a higher number of circuits present in the network.

As Drossel et al.[1, 2] have shown there are definite relationships between the number of attractors and the number of loops. Qualitatively summarizing, the number of structural circuits provides a lower bound on the number of possible attractors.

It therefore gives insight into the controversy over the number of attractors in random boolean networks to consider the exactly enumerated number and distribution of circuits in the underlying networks.

Experimental Results

0 10 11 6 13 3 4 15 0	1 8 4 13 12 1
0 10 11 6 13 12 1 8 0	1 8 4 13 12 1
0 10 11 6 13 12 1 8 4 15 0	3 3
0 10 11 6 13 12 1 8 0	3 4 13 3
0 10 11 6 13 12 1 8 4 15 0	3 6 13 3
0 10 11 6 13 15 0	6 13 12 10 11 6
0 14 11 6 13 3 4 15 0	6 13 12 14 11 6
0 14 11 6 13 12 1 8 0	8 8
0 14 11 6 13 12 1 8 4 15 0	9 9
0 14 11 6 13 12 1 8 0	12 12
0 14 11 6 13 12 1 8 4 15 0	
0 14 11 6 13 15 0	

FIGURE 2: 22 Circuits found in the network shown in figure 1 which has 16 nodes and 32 arcs and allows self-arcs. Note there are repeated circuits due to the presence of a multiple-arc connecting nodes 12 and 1.

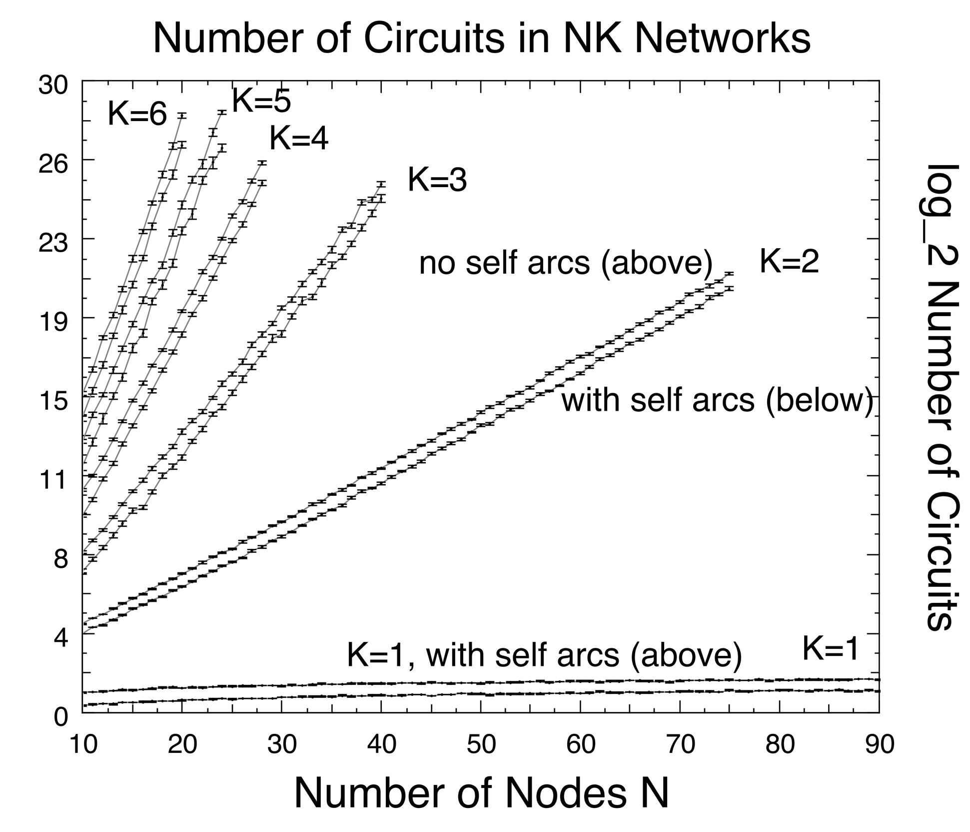

FIGURE 3: Growth in Number of Circuits in NK model with network size N

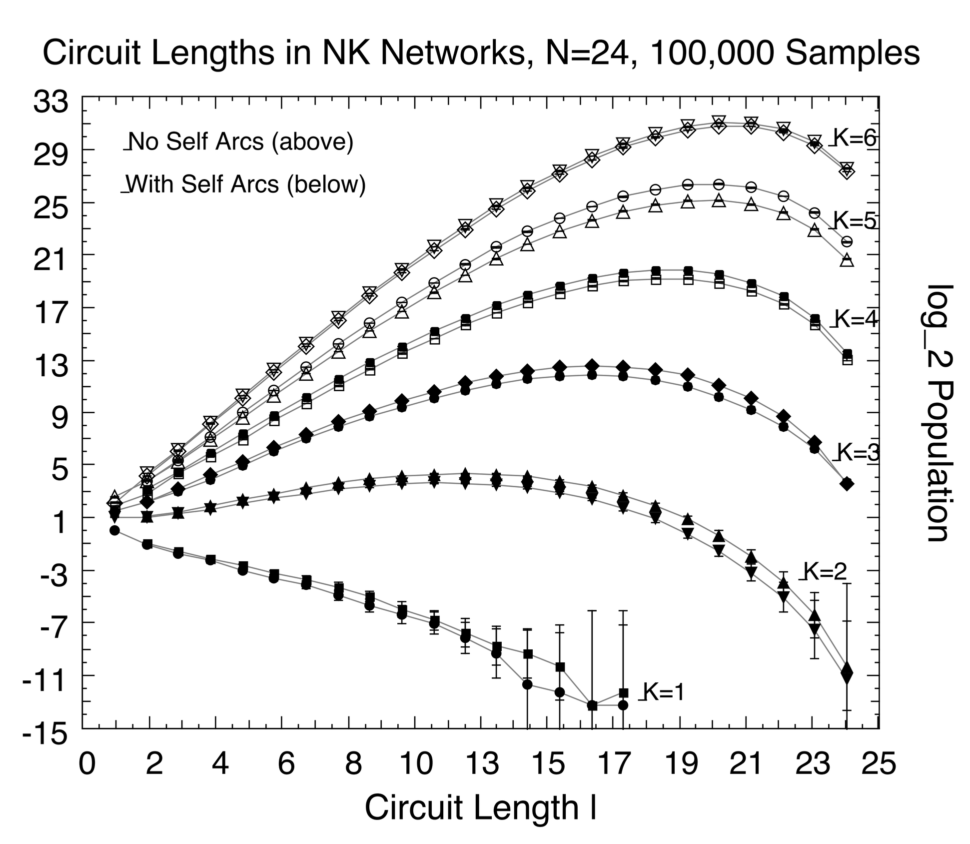

FIGURE 4: Distribution of Circuit Lengths in NK networks of size $N = 24$

Conclusions

A least-squares fit of figure 3 reveals that the number of circuits or loops varies as $N_L \approx A_L e^{bN}$.

As figure 4 shows, for $K > 1$ there will be circuits of lengths up to the Hamiltonian circuit length of $L_H \equiv N$, with a modal value at some lesser length that increases with K. For the case of $K = 1$ however, the modal length is always unity and the maximum circuit length is truncated (perhaps only in the limit of large N?) to $N/2$.

We have explored several of the structural properties of the NK-network model and have found that the number of circuits grows faster than any power law with network size[3]. This confirms that the number of attractor loops in Random Boolean Networks should also grow faster than any power law.

We have identified some intriguing structural behaviour between the values $1 < K < 2$ where the circuit length distribution function exhibits a transition from exponential decay to growth towards a non-unit modal value. Our data appears to show that the network self-edges or RBN nodes with self-inputs have a decisive role to play in influencing the location of the phase transition and hence the number of circuits and hence attractors present.

The structural components, which for $K > 1$, are completely dominated by the giant component and are insensitive in number to the presence or absence of self-edges. For $K = 1$ however, disconnected components of sizes up to half the network size are present, and not just monomers. For $K = 1$ the self-edges dominate the circuit size distribution and are the most prevalent loop type present.

We are investing more computational effort into studies of higher K values to investigate the exponent dependence on K for the number of circuits[4].

More information on our complex networks research projects is available from the web site: `www.massey.ac.nz/~kahawick/complex/networks.html`.

References

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